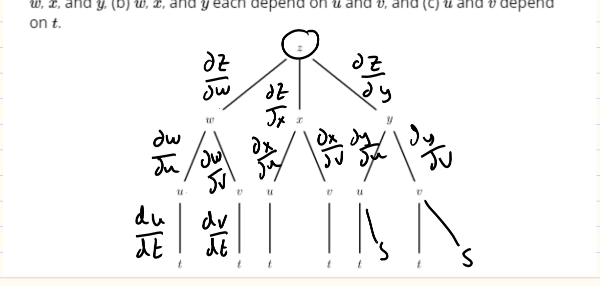
Meeting 8 Math 22

Reading Debrief

- Discuss reading on the Chain Rule w/ your group.
- Are there questions we should address?

Questions:

a. Figure 10.5.4 shows the tree diagram we construct when (a) z depends on w, x, and y, (b) w, x, and y each depend on u and v, and (c) u and v depend on t.



$z = f(w, x, y)$
$w(u, v)$
$x(u, v)$
$y(u, v)$
$u(t)$
$v(t)$

Find
$$\frac{dz}{dt} = \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}.$$

Definition 10.6.2 Let $f(x,y)$ be a function and $u = \langle u_1, u_2\rangle$ a unit vector. The directional derivative of f at (x_0, y_0) in the direction of u is the limit

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+u_1h, y_0+u_2h) - f(x_0,y_0)}{h}$$

whenever this limit exists.

$D_u f(x,y)$ measure the instantaneous rate of change of f per unit increase in the direction of u. It also measures the slope of the tangent to the curve obtained by intersecting the graph of f with the plane that:

- Contains the line through (x_0, y_0) with direction u
- is perpendicular to the plane $z=0$.

Section 10.6.2 Computing $D_u f(x_0, y_0)$

Let $f(x,y)$ be a function, $u = \langle u_1, u_2\rangle$ a unit vector, and (x_0,y_0) a point. We can compute $D_u f(x_0,y_0)$ using the chain rule. Parameterize the line through (x_0,y_0) in the direction of u as follows:

$$x(t) = x_0 + u_1 t \qquad y(t) = y_0 + u_2 t.$$

Compose $f(x,y)$ with the line (Then $f(x(t), y(t))$ is the blue curve from GeoGebra). Then

$$\begin{aligned} D_u(x_0,y_0) &= \lim_{h\to 0}\frac{f(x_0+u_1h, y_0+u_2h) - f(x_0,y_0)}{h} \\ &= \lim_{h\to 0}\frac{f(x(h),y(h)) - f(x(0),y(0))}{h} \\ &= \frac{d}{dt} f(x(t),y(t))\Big|_{t=0} \\ &= \frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt}\Big|_{t=0} + \frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}\Big|_{t=0} \\ &= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 \\ &= \underbrace{\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle}_{\nabla f(x_0,y_0)} \cdot u. \end{aligned}$$

"The gradient"

To summarize:

The directional derivative of f at (x,y) in the direction of the unit vector u is given by

$$D_u f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot u.$$

Activity 10.6.2

- Complete Activity 10.6.2 and discuss w/ your group.
- Discuss as a class.

a. $f_x(x,y) = 3y - 2xy^3$
$f_y(x,y) = 3x - 3y^2x^2$

b. $D_i f(x,y) = f_x(x,y)$
$D_j f(x,y) = f_y(x,y)$

Conclusion: the partial derivatives are a special case of the directional derivatives.

c. 1. Find a unit vector $\frac{v}{|v|} = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle$.

2. $D_{\frac{v}{|v|}} f(1,-1) = \langle -1, 0\rangle \cdot \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle = -\frac{2}{\sqrt{13}}.$

Section 10.6.3 The Gradient

Definition Let $f(x,y)$ be a function. The gradient of f at (x_0,y_0) is the vector

$$\nabla f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle.$$

The ∇ is pronounced "del". With this new definition, the formula for the directional derivative becomes

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0)\cdot u.$$

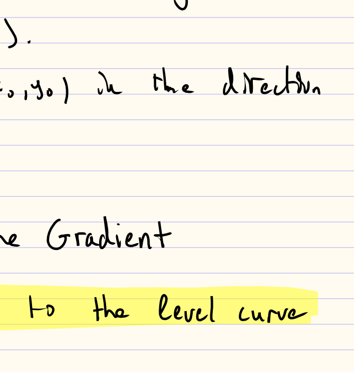
Activity 10.6.3

- Complete Activity 10.6.3 and discuss w/ your group.
- Class discussion.

a. $\nabla f(x,y) = \langle 2x, -2y\rangle$

b.
- $(x_0,y_0) = (2,0)$
- $(x_0,y_0) = (0,2)$
- $(x_0,y_0) = (2,2)$
- $(x_0,y_0) = (2,1)$
- $(x_0,y_0) = (-3,2)$
- $(x_0,y_0) = (-2,-4)$
- $(x_0,y_0) = (0,0)$

$\nabla f(2,0) = \langle 4, 0\rangle$
$\nabla f(0,2) = \langle 0, -4\rangle$
$\nabla f(2,2) = \langle 4, -4\rangle$
$\nabla f(2,1) = \langle 4, -2\rangle$
$\nabla f(-3,2) = \langle -6, -4\rangle$
$\nabla f(-2,-4) = \langle -4, 8\rangle$
$\nabla f(0,0) = \langle 0, 0\rangle$



c. The gradient vector $\nabla f(x_0,y_0)$ is orthogonal to the level curve through (x_0,y_0).

d. The function increases at (x_0,y_0) in the direction of the gradient $\nabla f(x_0,y_0)$.

Section 10.6.4 Direction of the Gradient

The gradient $\nabla f(x_0,y_0)$ is normal to the level curve through (x_0,y_0).

Proof. Let (x_0,y_0) is a point on the level curve. Let u be a unit vector $\perp$ to $\nabla f(x_0,y_0)$. Then

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0)\cdot u = 0.$$

The function f is not changing in the direction of u. Therefore u is tangent to the level curve through (x_0,y_0).

More generally, the gradient $\nabla g(x_0,y_0,z_0)$ of a function g of three variables is normal to the "level surface" through (x_0,y_0,z_0) defined by an equation of the form $c = g(x,y,z)$.

We can exploit this to derive the equation of a tangent plane to a graph. Let $f(x,y)$ be a fnc of 2 variables. We find the tangent plane at (x_0,y_0). Consider $g(x,y,z) = f(x,y) - z$. The level surface $0 = g(x,y,z)$ is just the graph of $f(x,y)$. By the above, $\nabla g(x_0,y_0,z_0)$ is normal to the graph of f.

$$\nabla g(x_0,y_0,z_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle.$$

therefore the eq. of the plane is

$$\langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0.$$

Activity 10.6.4

- Complete Activity 10.6.4 and discuss w/ your group.
- Class discussion.

a. $D_u f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot u = |\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle|\cdot \underbrace{|u|}_{=1}\cos\theta$

b. If $\theta = 0$, $\cos\theta = 1$. Since $0 \le \theta \le \pi$ and $-1 \le \cos\theta \le 1$ and $|\langle f_x, f_y\rangle| \ge 0$, choosing $\theta = 0$ maximizes the quantity.

c. When $\theta = 0$, the vector u is parallel to $\nabla f(x_0,y_0)$. Therefore, $\nabla f(x_0,y_0)$ points in the direction of steepest ascent of the graph of f.

d. $-\nabla f(x_0,y_0)$ is the direction of steepest descent.

e. The vectors

$$u = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

$$v = \frac{-\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

are unit vectors in the direction of steepest ascent/descent. We require that $|\nabla f(x_0,y_0)| \neq 0$.

What is the instantaneous rate of change in the direction of steepest ascent? We need to compute $D_u f(x_0,y_0)$ w/ u as above:

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|} = |\nabla f(x_0,y_0)|$$